ARML Inequalities Lecture

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Inequalities are one of the few topics that shows up during ARML that has not been covered in about five seperate lectures this year; when they do show up, you may wish to be able to solve them. They show up fairly often in the Mandelbrot competition, and occasionally in the AMC/AIME/USAMO. As it turns out, there are very few inequalities you actually have to know for these competitions (excepting, perhaps, the USAMO), and these basic inequalities will be covered in this lecture.

1 The Trivial Inequality

The trivial inequality simply states that for some number $x \in \mathbb{R}$,

$$x^2 \geq 0 \tag{1}$$

How is this useful to us? Consider the following:

$(x-y)^2 \geq 0 \implies x^2 - 2xy + y^2 \geq 0 \implies x^2 + 2xy + y^2 \geq 4xy \implies (\frac{x+y}{2})^2 \geq xy \implies \frac{x+y}{2} \geq \sqrt{xy}$ for non-negative x and y.

What we have just proven is that the arithmetic mean is greater than or equal to the geometric mean of two non-negative numbers. And all we used was the fact that the square of a real number is always non-negative! This brings us to the next inequality, which is the one you will be using in about 90% of the problems you do.

2 RMS-AM-GM-HM

This inequality states that for non-negative numbers $a_1, a_2, \cdots, a_n$,

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \tag{2}$$

That is, the root mean square is greater than or equal to the arithmetic mean, which is greater than or equal to the geometric mean which is greater than or equal to the harmonic mean of some group of non-negative numbers. In general, equality holds when $a_1 = a_2 = \cdots = a_n$. Let's see how we can use this by solving a fairly simple problem.

Problem: If $2w + 3x + 5y + 7z = 4$, and $w, x, y, z \geq 0$, find the maximum value of $wxyz$.

Solution: Plugging the given values into AM-GM, we have

$\frac{2w+3x+5y+7z}{4} = 1 \geq \sqrt[4]{(2w)(3x)(5y)(7z)} = \sqrt[4]{210wxyz}$. The maximum occurs when these are equal, so $wxyz_{max} = \frac{1}{210}$.

This can also be extended to the power-mean inequality, which states that for some numbers p and q such that $p > q$,

$$\sqrt[p]{\frac{a_1^p + a_2^p + \cdots + a_n^p}{n}} \geq \sqrt[q]{\frac{a_1^q + a_2^q + \cdots + a_n^q}{n}} \tag{3}$$

If we plug in some values of p or q, we see that $p = -1$ gives us the harmonic mean, $p = 1$ gives us the arithmetic mean, and $p = 2$ gives us the root mean square. As it turns out, you can use some very fancy math to show that the geometric mean corresponds to $p = 0$, but that involves limits, which are not the subject of this lecture.

3 The Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality, or often just the Cauchy inequality, states that for numbers $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$,

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \tag{4}$$

Proof. Consider a polynomial $\sum_{i=1}^{n}(x_i z + y_i)^2 = \sum_{i=1}^{n}(x_i^2 z^2 + 2x_i y_i z + y_i^2)$. By the trivial inequality, the polynomial is always greater than or equal to 0, so all of its roots will be either double or imaginary. That is, the discriminant of the polynomial is less than or equal to 0. Thus we have $(\sum_{i=1}^{n} 2x_i y_i)^2 - 4(\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2) = 4((\sum_{i=1}^{n} x_i y_i)^2 - (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)) \le 0 \implies (\sum_{i=1}^{n} x_i y_i)^2 \le (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)$, which is the Cauchy-Schwarz Inequality.

If we examine this proof, we can also see where equality occurs. This happens when the polynomial is 0, and due to the trivial inequality, each term is greater than or equal to 0. Thus, each term must equal 0, so $z = -\frac{y_i}{x_i}$. But z must be the same in all of the terms, so the equality condition is that $\frac{y_1}{x_1} = \frac{y_2}{x_2} = \cdots = \frac{y_n}{x_n}$.

Problem: (AoPS vol. 2) Prove that $1^2 + 2^2 + \cdots + n^2 \ge \frac{(1+2+\cdots+n)^2}{n}$ for all integers $n \ge 1$.

Solution: Multiplying by n, we have $(1^2 + 2^2 + \cdots + n^2)(n) \ge (1 + 2 + \cdots + n)^2$. We suspect that Cauchy's inequality is somehow involved (if not because we see sums of squares, then perhaps because this problem is in the Cauchy's section). To make this look like Cauchy's, we note that $n = (1^2 + 1^2 + \cdots + 1^2)$, so we have $(1^2 + 2^2 + \cdots + n^2)(1^2 + 1^2 + \cdots + 1^2) \ge ((1)(1) + (2)(1) + \cdots + (n)(1))^2$ which is true by the Cauchy-Shwarz Inequality.

4 Rearrangement

This inequality states that if $x_1, x_2, \cdots, x_n$ and $y_{1,2}, \cdots, y_n$ are similarly sorted,

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \le x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_n y_{\sigma(n)} \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \quad (5)$$

where σ is a permutation of $\{1, 2, \ldots, n\}$.

5 Calculus

As it turns out, you can often get around inequalities by bashing them with calculus (especially when the question is asking for a minimum or maximum). However, doing so is typically considered inelegant, and in proof based contests (i.e. the USAMO), the graders will often take every opportunity to take off points from a calculus based proof. That being said, calculus is a very useful tool when you don't feel like being clever.

6 General Tips

1. Usually you will be dealing with non-negative numbers, but remember that when you multiply or divide by a negative the inequality will flip (i.e. $>$ becomes $<$, $\leq$ becomes $\geq$, etc).

2. While doing a proof involving inequalities, if an inequality seems weak, you can sometimes prove a stronger bound instead (which can sometimes be easier to prove).

3. Try to make all of the terms in an inequality look similar, perhaps by using dummy variables (for example, if you're seeing a lot of $'2x'$s and $'3y'$s, you could make the substitutions $a = 2x$ and $b = 3y$. You can also do this for uglier expressions).

7 Problems

1. If $a, b > 0$, find the minimum value of $\frac{a}{b} + \frac{b}{a}$ (the answer is pretty easy to guess, but do this rigorously)

2. (BHamrick) There are three envelopes, one with \$1 bills, one with \$5 bills, and one with \$10 bills. You are allowed to take three bills from one envelope, two from another, and one from the last. What is the maximum amount of money you can obtain from the envelopes?

3. Use the trivial inequality to prove $RMS \geq AM$, $RMS \geq GM$, $RMS \geq HM$, $AM \geq HM$, and $GM \geq HM$ for two variables.

4. (JGeneson) Find, without calculus, the minimum value of $x^{\ln x+1}$.

5. Prove that $n^2 \leq (a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n})$.

6. (ARML 1987) If $a, b, c > 0$ and $a + b + c = 6$, show that $(a + \frac{1}{b})^2 + (b + \frac{1}{c})^2 + (c + \frac{1}{a})^2 \geq \frac{75}{4}$.