The Geometry of Circles

Bruce Sun and Kee Young Lee

October 3, 2007

1 Angles and Arcs

A central angle is an angle whose vertex lies on the center of a circle. For any given central angle, its measurement is equivalent to the measurement of the arc that it subtends. So if we were given a central angle with a measurement of θ radians, we would define the measurement of the intercepted arc to be θ as well. Note that the arc measurement of the entire circle is 2π.

But what if the vertex of the angle is not on the center? We can still use the intercepted arcs to calculate the angle measure. For the case when vertex P is inside the circle, and the angle is defined by two chords that intercept arcs with measure α and β, the angle measure $\theta = \frac{\alpha+\beta}{2}$. Similarly, if P were outside of this circle, and instead we had two secants intercepting arcs with measure α and β ($\beta > \alpha$), then $\theta = \frac{\beta-\alpha}{2}$. Exercise:*Prove to yourself that the measure of an inscribed angle intercepting an arc of measure θ is $\frac{\theta}{2}$*

2 Power of a Point

The Power of a Point theorem can be broken down into two cases: one where P is inside the circle, and one where P is outside the circle. For the case where P is inside the circle, Power of a Point states that if AB and CD are chords of the circle intersecting at P, then $(AP)(PB) = (CP)(PD)$. For the case where P is outside the circle, Power of a Point states that if AB and CD are secants that intersect at P, then $(AP)(PB) = (CP)(PD)$. Notice that for the second case, when A and B are coincident (the resultant line is now tangent to the circle), $AP = PB$.

3 Inscriptions and Circumscriptions

Given a triangle ABC, if we construct the perpindicular bisectors of AB,BC, and AC, then the bisectors are concurrent at the circumcenter of the triangle. Because the circumcenter happens to be equidistant from A,B, and C. We denote this distance as R, or the circumradius. The circle with radius R centered at the circumcenter is known as the circumcircle. R is most often used with the Extended Law of Sines (worthy of another lecture). On the other hand, if we construct angle bisectors from each angle, they will also be concurrent. This point of concurrency is known as the incenter, which is equidistant from sides AB,BC, and AC. Denoting the distance as r, the incircle is the circle radius r with center at the inradius. This is particularly useful with the area formula $K = rs$, where $s = \frac{a+b+c}{2}$.

4 Cyclic Quadrilaterals

Circle geometry becomes much more interesting once we start inscribing and circumscribing stuff with circles. A quadrilateral is cyclic if it can be inscribed in a circle. Unlike triangles, not all quadrilaterals can be inscribed in a circle. But for the ones that can, many very interesting (and useful) properties arise. Let ABCD be a quadrilateral. Then ABCD is cyclic iff angles A and C are supplementary. This is fairly simple to prove. If we think of a cyclic quadrilateral as two inscribed angles, then the sum of the angle measures is equal to half of the sum of the arc measures. As an exercise, try to prove the converse. Taking the same quadrilateral, suppose we constructed chords AC and BD intercecting at P. Note the similar triangles:$\triangle ABP \sim \triangle CDP$, $\triangle ACP \sim \triangle BDP$. Also, because of the nature of the cyclic quadrilateral, we also have power of a point with the intersecting chords. These are often telltale signs of cyclic quadrilaterals, so watch for them. Two of the most important cyclic quadrilateral theorems: **Ptolemy's Theorem** Let cyclic quadrilateral ABCD have sides of (in rotational order) length a, b, c, and d, and diagonals of length e and f, then $ac + bd = ef$. **Brahmagupta's Theorem** Let a quadrilateral have sides of length a, b, c, and d and angles A, B, C, and D, and let s be the semiperimeter, so $s = \frac{a+b+c+d}{2}$. Then $K = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2(\frac{A+C}{2})}$. What's interesting about the extended version of this theorem is that it is pretty much useless. In the case of cyclic quadrilaterals, however, note that the cosine term falls off, s we have $K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. This formula resembles Heron's formula quite a bit. In fact, letting $d = 0$ collapses the fourth side, giving us the area of a triangle in terms of a, b, and c.

5 Practice

1. Let ABC be a scalene right triangle, and P is the point on hypotenuse AC such that $\angle ABP = 45°$. Given that $AP = 5$ and $CP = 7$, find the area of ABC.

2. Let ABC be a triangle such that $\angle BAC = 2\angle BCA$. Given that $AB = 2$ and $BC = 3$, find AC.

3. Let ABC be an equilateral triangle with incenter I. Let P be a point on the opposite side of line BC as I. Given that $PI = AI$, $PB = 3$, and $PC = 4$, find PA.

4. Let ABC be an equilateral triangle with incenter I. Let P be a point on the opposite side of line BC as I. Given that $PI = AI$, $PB = 3$, and $PC = 4$, find the area of quadrilateral $ABPC$.

5. P is a point inside triangle ABC, and lines AP, BP, CP intersect the opposite sides BC, CA, AB at points D, E, F, respectively. Given that $\angle APB = 90°$, $AC = BC$, $AB = DB$, $BF = 1$, $BC = 5$, find AF.