TJUSAMO Practice 13 - Functional Equations

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This lecture will be mostly problem based. Functional equations = cleverness.

1 Common Techniques

Here is a list of things to remember to try to use with functional equations. I took this straight from Reid Barton.

1. Guess the answers and use them.
2. Plug in 0 and 1 or other small cases.
3. Make things equal. This could be as simple as $x = y$ or other much less obvious stuff.
4. Repeated application. Applying the function to the whole thing again.
5. Do a lot of casework based on certain values.
6. The idea of a surjective or injective function can force a certain value.
7. Build up a lot of values, show they're dense, and then use continuity.

2 Practice Problems

These are all compiled from different MOSP sources.

1. (Cauchy's Functional Equation) Find all continuous solutions to $f(x+y) = f(x)+f(y)$.
2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all $x, y \in \mathbb{R}$.
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y, z, t

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz)$$

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ which satify $f(m + f(n)) = f(f(m)) + f(n)$ for all $m, n \in \mathbb{Z}$.
5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.
6. Let $f : [0, 1] \to \mathbb{R}$ be a function such that $f(1) = 1$, $f(x) \geq 0$, and if $x, y, x + y$ all lie in $[0, 1]$, then $f(x + y) \geq f(x) + f(y)$. Prove that $f(x) \leq 2x$ for all $x \in [0, 1]$.
7. Find all functions $f : \mathbb{R} \to [0, \infty)$ such that for all x, y real:
$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$
8. Find all polynomials $p(x)$ such that for all x:
$$(x - 16)p(2x) = 16(x - 1)p(x)$$
9. Determine all polynomials $P(x)$ with real coefficients such that
$$(x^3 + 3x^2 + 3x + 2)P(x - 1) = (x^3 - 3x^2 + 3x - 2)P(x)$$
10. Find all pairs of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
$$f(x + g(y)) = xf(y) - yf(x) + g(x)$$
11. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,
$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$