TJVMT POTW

Set #1 Solutions

11/07/07 - 11/14/07

1S. Find the number of integers n less than 1000 such that n is equal to the sum of the fourth powers of its digits. For example, the sum of the fourth powers of the digits of 233 is 178, so 233 would not be such an integer.

Answer: 2
Solution: Since fourth powers are nonnegative, n is nonnegative, so let $n = abc$. Then we have the diophantine equation $a^4 + b^4 + c^4 = 100 * a + 10 * b + c$. Let $M = max(a, b, c)$. If $M \geq 6$, then the LHS is at least $6^4 = 1296$, but the RHS is at most 999. So $M \leq 5$. If $M = 5$, the LHS is at least $5^4 = 625$, so $a \geq 6$, so $M \geq 6$, contradiction, so $M \leq 4$. Since $x^4 \cong 1 \pmod 5 \equiv 5 \nmid x$ and $0 \leq c \leq M \leq 4$, c equals the number of nonzero digits in n. If $c = 0$, then $n = 0$, which is a trivial solution to the equation. Now assuming $c > 0$, c is one more than the number of nonzero elements of a, b, so try all 25 cases $0 \leq a, b \leq 4$, and see that only $a = b = 0$ generates a solution. Thus we have two solutions to our diophantine equation, $n = 0, 1$.
P.S. Disregarding the condition $n < 1000$, we have many possible integers n that satisfy the property. Finding them, however, involves more guesswork than anything else.

2P. I write two positive integers and two negative integers on a blackboard. Every minute afterwards, I replace the four integers w, x, y, z on the blackboard with $w-x, x-y, y-z, z-w$. Prove that after a day, at least one of the numbers will be larger than a googol.

Proof: Consider the sum of the squares of the four integers on the blackboard. I claim that this number doubles every minute. Note that after one minute, the four integers on the blackboard will always sum to zero. Also note that the numbers will never all be zero. Assume that they were all zero at sum point. Take the earliest minute at which they were all zero. This cannot be the first minute or the second minute. The minute beforehand, all the numbers were the same, but since they sum to zero, they are all zero, which is a contradiction to our assumption. Next we do some algebra: $(w-x)^2 + (x-y)^2 + (y-z)^2 + (z-w)^2 = (w-x)^2 + (x-y)^2 + (y-z)^2 + (z-w)^2 + (w+x+y+z)^2 = 3(w^2+x^2+y^2+z^2) + 2wy + 2xz = 2(w^2 + x^2 + y^2 + z^2) + (w+y)^2 + (x+z)^2 \geq 2(w^2 + x^2 + y^2 + z^2)$. The minimum the value can be in the beginning is $1^2 + 1^2 + 1^2 + 1^2 = 4$. After a day, this value doubles at least 1440 times, so it is at least 2^{1442}. The largest of the four numbers will be at least the square root of one fourth of the value, which will be at least $\sqrt{\frac{2^{1442}}{4}} = 2^{720}$. Note that

$2^{720} > 2^{400} = 16^{100} > 10^{100}$, so one of the numbers will be much larger than a googol.
P.S. Obviously, a googol is not a strict bound. We have proved that 2^{720} is a lower bound, and the example $w = y = 0, x = 1, z = -1$ achieves this bound.

3P. Given any convex polygon P, prove that there is a hexagon completely inside P with over three quarters the area of P.

Proof(trivial details omitted): Let $\triangle ABC$ be the triangle in P with the greatest area. Draw three lines through A, B, C and parallel to BC, CA, AB, respectively, and let $A_1B_1C_1$ be the triangle formed by these three lines such that A, B, C are on lines B_1C_1, A_1C_1, A_1B_1, respectively. Clearly P is inside $\triangle A_1B_1C_1$. Then we draw lines parallel to the sides of $\triangle ABC$ to form a convex hexagon $U_aV_aU_bV_bU_cV_c$ containing P, such that U_a, U_b, U_c are on A_1C_1, A_1B_1, B_1C_1, respectively, and V_a, V_b, V_c are on A_1B_1, B_1C_1, C_1A_1, respectively, and each of the line segments U_aV_a, U_bV_b, U_cV_c contains points of P, three of such will be named A_0, B_0, C_0, respectively. I claim that hexagon $AC_0BA_0CB_0$ has area at least three quarters that of P.
Let x, y, z be the areas of $\triangle U_aBC, U_bCA, U_cAB$, respectively. Let S be the area of ABC, S_p the area of P, and S_1 the area of hexagon $AC_0BA_0CB_0$. Note that $\triangle A_1U_aV_a$ is similar to $\triangle A_1BC$ with ratio $r = \frac{S-x}{S}$, so the area of U_aV_aCB is $S(1-r^2) = 2x - \frac{x^2}{S}$. Combining similar formulas for U_bV_bAC and U_cV_cBA and summing, we get $4S_1 - 3S_p \geq S - 2(x + y + z) + \frac{x^2+y^2+z^2}{S} = \frac{S-x-y-z)^2}{S} \geq 0 \equiv S_1 \geq \frac{3S_P}{4}$. ■
P.S. After constructing the desired hexagon, which is the challenging part, the remaining algebra is straightforward.