

TJVMT POTW

Set #2 Solutions

11/14/07 - 11/21/07

1P. Prove that the area of any triangle whose vertices have integral coordinates is half of an integer.

Proof: Draw a box around the triangle with sides parallel to the axes. The area of the triangle is the area of the box minus the area of the triangles inside the box but outside the given triangle, whose legs have integer side lengths. This area is an integer minus half of an integer, which is half of an integer.

2S. The real number line between 0 and 1 is initially colored orange. It is first split into five equal segments, of which the second and fourth segments(not including their endpoints) are colored blue. Then each of the three orange segments left over is again split into five equal segments, of which the second and fourth segments(again, not including their endpoints) are colored blue. This is done to each of the remaining orange segments ad infinitum. What is the probability that a real number randomly and uniformly selected between 0 and 1 will eventually be colored blue by this process?

Answer: 1

Solution: At each step, $\frac{2}{5}$ of the remaining orange space is colored blue, so the size of the blue space is multiplied by $\frac{3}{5}$. The blue space thus becomes infinitesimally small, and the size of the orange space converges to 1.

3S. Let O be the origin, and X, Y, Z be points on the positive x-axis, y-axis, and z-axis, respectively, such that $\angle XYZ = 72^\circ$ and $\angle XZY = 49^\circ$. Let T be the plane of $\triangle XYZ$. Point P is a point in T such that OP is perpendicular to T . Find $\angle XPZ$.

Answer: 108°

Solution: It's well known that P is the orthocenter of $\triangle XYZ$. Therefore, $\angle XPZ = 180^\circ - \angle XYZ = 72^\circ$.