TJVMT POTW

Set #7 Solutions

12/19/07 - 1/9/08

1S. Four buildings are located at the vertices of a unit square. You are trying to connect these cities with a system of roads, which are represented by straight lines. What is the smallest length of road that can be used to connect all four cities?

Answer: $1 + \sqrt{3}$

Solution: Let E be the center of our square $ABCD$. By symmetry the roads must go through E, so we can split the path up into two paths, one which joins ABE and one which joins CDE. By Fermat, the shortest path which joins the three vertices of an acute triangle goes through the Fermat point of the triangle. Since our two triangles are congruent and are both isosceles right triangles, the Fermat point is easy to compute. Its distance from the edge is $\frac{1}{2\sqrt{3}}$, so the total path for one triangle is $\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} + (\frac{1}{2} - \frac{\sqrt{3}}{6})$, and multiply by two for the total path of the road system, which is $1 + \sqrt{3}$.

2P. Let $r_1, r_2, ..., r_n, c_1, c_2, ..., c_n$ be **distinct** positive integers, and let a grid with n rows and n columns be such that the ith row of the jth column is filled with the number $r_i + c_j$. Given that the product of the numbers in each row is constant, prove that the product of the numbers in each column is constant.

Proof: Unfortunately the word distinct was omitted from the problem statement, which made it false. The solution for the revised problem is as follows. Consider the polynomial $p(x) = \prod_{i=1}^{n} x + r_i - \prod_{i=1}^{n} x - c_i$. Note that for each j from 1 to n, $p(c_j) = \prod_{i=1}^{n} c_j + r_i = c$, for some constant c, so $p(x) = c$ for at least n values, but $p(x)$ has degree less than n, so $p(x) = c$ for all real x. Now for all values j from 1 to n, plug in $-r_j$ and get $c = \prod_{i=1}^{n} -r_j - c_i$, so the product of the numbers in each column is just $-c(-1)^n$.

3P. Given a triangle with side lengths a, b, c, prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$$

Proof: WLOG a is the longest side. Note that $a(b-c)^2(b+c-a)+b(a-b)(a-c)(a+b-c) \geq 0$ since each term is positive. But woah, this is the same as our desired result(by expansion). Note: There is a longer but more straightforward solution which involves the substitutions $x = b + c - a, y = a + c - b, z = a + b - c$ followed by Rearrangement.