TJ USAMO Practice 16

VMT Math Team

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"Which theorem is Botchy's theorem?" - M.Lee

1. (Multivariable Calculus Semester Exam '04) A function $f(x, y) = 2x^2 + y^2 - 6x + 8y - 9$ is defined for all real numbers x and y. Does f have a minimum? If so, what is it and where does it occur?

2. P is a point (a, b, c) on the sphere of radius 10 that is centered at the origin. Determine the maximum possible value of $a + 2b + 3c$.

3. (MOP '03) On a standard chessboard, a *dolphin* may make a *move* in one of three ways: translating down one square, left one square, or diagonally up and right (one square in each direction.) If a dolphin is placed in the bottom left corner of a chessboard, is it possible that after a sequence of moves the dolphin has traversed the board visiting each square exactly once? Prove your answer.

4. (7th Czech-Slovak-Polish Match) Show that for $n \geq 2$, if $a_1, a_2, \ldots a_n > 0$, then

$$\prod_{i=1}^{n} \left(a_i^3 + 1\right) \geq \prod_{i=1}^{n} \left(a_i^2 a_{i+1} + 1\right)$$

   where a_{n+1} is taken to represent a_1.

5. (China '03) ω_1, ω_2, and ω_3 are three complex numbers such that

$$\begin{aligned} \omega_1 + \omega_2 + \omega_3 &= 1 \\ \omega_1^2 + \omega_2^2 + \omega_3^2 &= 3 \\ \omega_1^3 + \omega_2^3 + \omega_3^3 &= 7 \end{aligned}$$

   Determine, with proof, the value of $\omega_1^{12} + \omega_2^{12} + \omega_3^{12}$.