TJ USAMO Practice 13 - Functional Equations

Varsity Math Team

January 27, 2005

1. (Cauchy) Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property that $f(x+y) = f(x) + f(y)$ for all real x and y. Show that f is a line that contains the origin. That is, show that $f(x) = x \cdot f(1)$.

2. $f : \mathbb{R} \to \mathbb{R}$ is a non-constant function that obeys $f(\alpha(x+y)) = f(x) + f(y)$ for all real x and y and some real constant α. Determine all possible values of α.

3. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that for all rational x and y,

$$\begin{aligned} f(x+y) - yf(x) - xf(y) &= f(x)f(y) - x - y + xy & (1) \\ f(x) &= 2f(x+1) + x + 2 & (2) \\ f(1) + 1 &> 0 & (3) \end{aligned}$$

4. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that $f(x+y) + f(x)f(y) = f(x) + f(y) + f(xy)$ $\forall x, y \in \mathbb{Z}$. (Query: Do all of the functions $g : \mathbb{Q} \to \mathbb{Q}$ with the property that $g(x+y) + g(x)g(y) = g(x) + g(y) + g(xy)$ $\forall x, y \in \mathbb{Q}$ necessarily satisfy $g(n) = f(n)$ for each integer n? What if $g : \mathbb{R} \to \mathbb{R}$ and $x, y \in \mathbb{R}$?)