# TJ USAMO Practice 6 - Cyclic Quadrilaterals

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Cyclic quadrilaterals come up very often so it is instructive to learn as much as possible about them. They are most often useful for angle chasing because as you will see, they involved a lot of angles. It is important to realize that very few problems will only require noticing a cyclic quadrilateral , more often, it could be one critical observation to get us closer to our goal. Before I tell you anything, why don't we make some observations about cyclic quadrilaterals?

## 1 What does a Cyclic Quadrilateral look like?

To answer this question why don't we try drawing one? Take a circle $\omega$ with center O. Inscribe a cyclic quadrilateral ABCD into it. Now draw the diagonals AC and BD. Let us make some observations. The first thing to remember is that the measure of an inscribed angle is half the measure of the central angle. Also, the measures of two inscribed angles that cut off the same arc are equal. Using these two facts combined with the knowledge that a whole circle constitutes 360°, we make two observations about the inscribed quadrilateral. They turn out being if and only if statements:

1. Convex quadrilateral ABCD is cyclic iff $\angle ABD = \angle ACD$.

2. Convex quadrilateral ABCD is cyclic iff $\angle ABC + \angle CDA = 180°$.

## 2 Some Theoroms

Usually we will want to use cyclic quadrilaterals to do angle chasing since they give us a lot of information about angles within the quadrilateral. To get information about other things there are a couple of other theoroms you should know.

- **Power of a Point** - Take a circle $\omega$ and a point P. Take any line through P that intersects the circle. The product of the distance from the point P to the first intersection and the distance from P to the other intersection is independant of the choice of line. If the line is tangent, we just take the tangent squared.

- **Ptolemy's Inequality** - Take a quadrilateral ABCD. We know that:

$$(AB)(CD) + (BC)(AD) \geq (AC)(BD)$$

With equality iff ABCD is cyclic.

- **Brahmagupta's Theorom** - Given a cyclic quadrilateral ABCD with side lengths a, b, c, and d.

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where s is the semiperimeter and K is the area of the quadrilateral.

# 3 Practice

1. Prove Power of a Point.

2. (Honsberger) Let ABC be an equilateral triangle inscribed inside of a circle. Then choose a point P on the circumscribed circle. Prove that the sum of the shorter two of PA, PB, and PC equals the third.

3. Prove the equality part of Ptolemy's inequality.

4. (MOSP) In triangle ABC, points S and T lie on AB, such that $\angle ACT = \angle BCS$. Points K and L are drawn such that AK is perpendicular to CK and BL is perpendicular to CL. CH is an altitude of the triangle and M is the midpoint of side AB. Prove that points H, M, K, and L are concyclic.