TJ USAMO Practice 8 - Pigeonhole

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Say you have 10 pigeons and only 9 holes. And say you have this incredible need to stuff those pigeons into those holes. Every single pigeon must be stuffed into a hole. In fact, we are playing a game and I win if there ends up being 2 pigeons in at least one of the holes. Do I win?

Of course I do!

And that was how the Pigeonhole Principle was born. It states that if I have n pigeons and m holes with $n > m$, I know that I must stuff at least one of the holes with two pigeons.

1 Generalizations

Equally intuitive are some generalizations of this principle:

- If we have n pigeons and m holes and we stuff the pigeons in the holes, then at least one of the holes has at least $\lceil \frac{n}{m} \rceil$ many pigeons in it.
- "Continuous pigeonhole" also applies. It is just as intuitive but is also very difficult to make rigorous. Use it anyways. It basically says if you try to stuff area or volume (fat pigeons), into a smaller area or volume there must be overlap. For example, you can't fit two squares of area 100 into a square of area of 150 without overlap.

2 Practice

1. Given an equilateral triangle with side length 1 and 5 points in it, prove that there must be 2 points that are within a distance of 1/2 of each other.
2. If $a_1, a_2, \cdots, a_n$ are integers, show that some non-empty subset of them has sum divisible by n.
3. (MOSP 2005) In a square grid of 169 points, 53 are marked. Prove that some 4 marked points form a rectangle with sides parallel to the grid lines.

4. (MOSP 2005) There are some circles inside a square with side 1. Suppose that the total circumference of the circles of 10. Prove that there is a line which can cut through 4 of those circles.

5. (Dirichlet) Let α be an irrational number. Show that there are infinitely many rational numbers $\frac{m}{n}$ where m and n are relatively prime integers, for which $|\alpha - \frac{m}{n}| < \frac{1}{n^2}$. Then prove the converse just for fun!

6. (MOSP 2005) Given any $n + 2$ distinct numbers from the first $3n$ positive integers, prove that it is always possible to find two numbers a and b from them such that $n < a - b < 2n$.