TJUSAMO Practice 14 - Geometry

PDiao06

April 2, 2006

Thankfully, according to Thomas Mildorf, geometry is low-tech on the USAMO. This means that most of those really crazy things you always hear about are not worth your time learning at this point if you don't already know them. Mainly, the big topics are similar triangles, cyclic quads, and power of a point. Other geometry that you have learned from short problems are probably sufficient to make most geometry at least accessible to you. Before even starting, I'll give you guys some tips I've received from people much better at geometry than I.

1 Tips From Some Masters

1.1 Master 1 - Zuming Feng aka the Zumbot

I'm pretty sure he didn't say these things, but they sound reasonable to me.

- Draw good diagrams. This means big ones, accurate ones, and bring your compass, ruler, protractor, graph paper, take advantage of this stuff.
- Make cyclic quads and parallel lines. They are handy for everything!
- Don't forget trig.
- Consider the area.
- Find the special point of concurrence.

1.2 Master 2 - Richard Rucccccyzzks

I think I mispelled that. It's a reflection of the fact that I can't pronounce that name. These are actually directly from him.

- Big diagrams (he says it too! Pictures tell you the answers that you have to prove. They tell you intermediate steps and also help you work backwards. It's the equivalent of trying small numbers in a number theory problem, it quickly tells you an angle is a right angle if every time you draw it, it looks like one!)

- Don't panic. (I don't know how that's gonna help.)
- Work in two directions. (absolutely crucial)
- Ask yourself what you haven't used yet when stuck.

2 Basic tool development

2.1 Similar triangles

There isn't much to say about them. Angle's the same and side ratios. It is surprising how darn useful they are and how noticing a clever similar triangle often finishes a problem. For fun, prove angle bisector theorem!

2.2 Circle stuff

1. Cyclic Quadrilaterals - go back to an earlier TJUSAMO and see the lecture on these. They are crucial for angle chasing and are very useful in other ways too such as Ptolemy's.
2. Angles - prove the intercepted angles stuff to yourself. All of them, no need to memorize, just prove it to yourself.
3. Power of Point? What, who needs that! Prove it and then promptly forget such a useless piece of junk existed. The proof gives you far more information than the theorem itself.
4. Radical Axis Theorem - I was just kidding about the power of a point thing, it is actually extremely useful. This theorem you might not know yet. The radical axis of two circles is the locus of points that has equal powers with respect to two different circles. It is always a line (this is proved by ugly coordinates). The radical axis theorem states that given three circles, the three radical axes always concur at one point or are parallel.

3 Practice

Here are some fun problems we can do together or you can do alone after this practice.

1. (USAMO 2003 Number 3) ABC is a triangle. A circle through A and B meets the sides AC, BC at D, E respectively. The lines AB and DE meet at F. The lines BD and CF meet at M. Show that M is the midpoint of CF iff $MB * MD = MC^2$.

2. (WOOT) WXYZ is a cyclic quadrilateral with diagonals that meet at Q. M is the midpoint of YZ. The circle through Q which is tangent to YZ at M meets WY at T and XZ at P. R is on XQ such that XR = ZP. S is on WQ such that $SR||WX$. Prove that WS = TY.

3. (WOOT) Line AB is tangent to circle O at point Y, with Y between A and B on the line. Point X is on circle O such that XY is a diameter. XA and XB meet the circle again at C and D, respectively. AD and BC meet the circle again at E and F, respectively. Prove that XE = XF.

4. Let ω be the circumcircle of triangle ABC, with orthocenter H. Let altitude AD intersect ω at A'. Prove that $HD = DA'$.

5. Three congruent circles pass through a common point P. The other points of intersection are A, B, and C. Prove that P is the orthocenter of triangle ABC and prove that the circumcircle of triangle ABC has the same radius as the three original circles.

6. (WOOT) A, B, C, D are circles such that A and B touch externally at P, B and C touch externally at Q, C and D touch externally at R, and D and A touch externally at S. A does not intersect C, and B does not intersect D. Show that PQRS is cyclic.

7. (IMO Shortlist 2004 G1) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.

8. (MOSP 2005) Let ABC be a triangle. Suppose that the circle through C tangent to AB at A and the cricle through B tangent to AC at A have different radii, and let D be their second intersection. Let E be the point on the ray AB such that $AB = BE$. Let F be the second intersection of the ray CA with the cricle through A,D,E. Prove that $AF = AC$.

9. (MOSP 2005) Line AB is tangent to Circle ω_1 at B. Let C be a point not on ω_1 such that segment AC meets ω_1 at two distinct points. Circle ω_2 is tangent to line AC and ω_1 at C and D, respectively, such that D and B are on opposite sides of line AC. Prove that the circumcenter of triangle BCD lies on the circumcircle of triangle ABC.