TJUSAMO Practice #16: Yrtemoeg

HMao

January 29th, 2007

You might think geometry is all nice and clean, but think again. There are many ways to deal with geometry that you might not have thought of before. For now, we will deal with the world of transformations, especially inversion. The ideas here will be less elementary than before, so you will be expected to already know the basics.

1 Inversion

Inversion is difficult to understand without the a several thousand word explanation(or several diagrams), but the basic idea is that given a circle ω with center O and radius r, an inversion with respect to a ω will map a point P to another point Q on the ray OP such that $OP * OQ = r^2$. Some notable observations about inversion are that it maps lines and circles into lines and circles, it generally maps other types of loci into weird shapes, it has a period of two, the only points that are mapped to themselves are the points on the circle, and the center is mapped to the point at infinity. Note that inversion often goes hand in hand with complex numbers, so using complex numbers in a geometry problem will allow inversion about the unit circle to be easily manipulated with the $T(z) = \frac{1}{\bar{z}}$ transformation. Also, inversion can be extended to higher dimensions.

Somewhat Traditional Example: You are on a beach playing beach volleyball and a tsunami is quickly approaching. How do you escape the tsunami?

Solution: Hurry inside the beach ball and perform an inversion with respect to it. Now the tsunami is trapped inside the beach ball, and you are outside.

2 Other Transformations

These are fairly self-explanatory and often can be intuitively seen, but it might be useful to mention the transformation in order to make the proof more rigorous.

-Translation

-Rotation

-Reflection

-Expansion/Enlargement/Homothety/Homothecy

Here is some more useful vocabulary:

-isometry/congruence: lengths and angles are preserved

-similarity: angles are preserved

-dilation/dilatation: transforms lines into parallel lines

-half-turn: 180 degree rotation

-central dila(ta)tion(same as expansion): a dila(ta)tion that is not a translation

-spiral similarity: central dilation/dilatation plus rotation about the same point

3 Problems

1. {1} A bird is on the top of a 200 meter tall building and wants to land on the ground, then fly to the top of a 300 meter tall building. If the two buildings are perpendicular to the ground and 1200 meters apart, what is the shortest distance in meters that the bird must travel?

2. {2} Four squares are constructed outside a parallelogram such that the one of the sides of each of the squares is a respective side of the parallelogram. Prove that the centers of the four squares form another square.

3. {3} (IMO 1981) Three congruent circles share a common point P and lie inside $\triangle ABC$, which has incenter I and circumcenter O. If each side of $\triangle ABC$ is tangent to two of the three circles, prove that the points I, O, P are collinear.

4. {4} (IMO SL 1993) $\triangle ABC$ has incenter I. A circle is internally tangent to the circumcircle of $\triangle ABC$, and it is also tangent to sides AB and AC at X and Y, respectively. Prove that I is the midpoint of XY.

5. {5} (MOP 2005) Given a cyclic quadrilateral $ABCD$, let AD and BC intersect at E such that C is between B and E, let AC and BD intersect at F, let M be the midpoint of CD, and let N be a point on the circumcircle of $\triangle ABM$ other than M such that $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear.