

Inequalities: Advanced Techniques

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1 Review

Last time we learned about **Weighted Power Mean**, **Jensen**, **Schur**, **Nesbitt**, **Rearrangement**, **Chebyshev**, **Majorization**, and **Muirhead**, as well as the **Trivial Inequality**.

We also learned about the techniques of **Dumbassing**, **Homogenizations**, **Smoothing**, and **Substitutions**. As a review, here are the theorems from last time:

Throughout this lecture, we refer to *convex* and *concave* functions. Write I and I' for the intervals $[a, b]$ and (a, b) respectively. A function f is said to be convex on I if and only if $\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y)$ for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Conversely, if the inequality always holds in the opposite direction, the function is said to be concave on the interval. A function f that is continuous on I and twice differentiable on I' is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$ (Concave if the inequality is flipped.)

Let $x_1 \geq x_2 \geq \dots \geq x_n$; $y_1 \geq y_2 \geq \dots \geq y_n$ be two sequences of real numbers. If $x_1 + \dots + x_k \geq y_1 + \dots + y_k$ for $k = 1, 2, \dots, n$ with equality where $k = n$, then the sequence $\{x_i\}$ is said to *majorize* the sequence $\{y_i\}$. An equivalent criterion is that for all real numbers t ,

$$|t - x_1| + |t - x_2| + \dots + |t - x_n| \geq |t - y_1| + |t - y_2| + \dots + |t - y_n|$$

We use these definitions to introduce some famous inequalities.

Theorem 1 (Trivial Inequality) For all real numbers x , $x^2 \geq 0$.

Theorem 2 (Jensen) Let $f : I \rightarrow \mathbb{R}$ be a convex function. Then for any $x_1, \dots, x_n \in I$ and any nonnegative reals $\omega_1, \dots, \omega_n$ with positive sum,

$$\omega_1 f(x_1) + \dots + \omega_n f(x_n) \geq (\omega_1 + \dots + \omega_n) f\left(\frac{\omega_1 x_1 + \dots + \omega_n x_n}{\omega_1 + \dots + \omega_n}\right)$$

If f is concave, then the inequality is flipped.

Theorem 3 (Weighted Power Mean) If x_1, \dots, x_n are nonnegative reals and $\omega_1, \dots, \omega_n$ are nonnegative reals with a positive sum, then

$$f(r) := \left(\frac{\omega_1 x_1^r + \dots + \omega_n x_n^r}{\omega_1 + \dots + \omega_n}\right)^{\frac{1}{r}}$$

is a non-decreasing function of r , with the convention that $r = 0$ is the weighted geometric mean. f is strictly increasing unless all the x_i are equal except possibly for $r \in (-\infty, 0]$, where if some x_i is zero f is identically 0. In particular, $f(1) \geq f(0) \geq f(-1)$ gives the AM-GM-HM inequality.

Theorem 4 (Hölder) Let a_1, \dots, a_n ; b_1, \dots, b_n ; \dots ; z_1, \dots, z_n be sequences of nonnegative real numbers, and let $\lambda_a, \lambda_b, \dots, \lambda_z$ positive reals which sum to 1. Then

$$(a_1 + \dots + a_n)^{\lambda_a} (b_1 + \dots + b_n)^{\lambda_b} \dots (z_1 + \dots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \dots z_1^{\lambda_z} + \dots + a_n^{\lambda_a} b_n^{\lambda_b} \dots z_n^{\lambda_z}$$

This theorem is customarily identified as Cauchy when there are just two sequences.

¹Thanks to Thomas Mildorf for writing the lecture on Inequalities, available online at <http://web.mit.edu/tmildorf/www/Inequalities.pdf>, as well as for generously providing me with the .tex file for said lecture so that I did not have to recopy all of the theorems. In addition, many example problems are also taken from his lecture.

Theorem 5 (Rearrangement) Let $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$ be two nondecreasing sequences of real numbers. Then, for any permutation π of $\{1, 2, \dots, n\}$, we have

$$a_1b_1 + a_2b_2 + \dots + a_nb_n \geq a_1b_{\pi(1)} + a_2b_{\pi(2)} + \dots + a_nb_{\pi(n)} \geq a_1b_n + a_2b_{n-1} + \dots + a_nb_1$$

with equality on the left and right holding if and only if the sequence $\pi(1), \dots, \pi(n)$ is decreasing and increasing respectively.

Theorem 6 (Chebyshev) Let $a_1 \leq a_2 \leq \dots \leq a_n$; $b_1 \leq b_2 \leq \dots \leq b_n$ be two nondecreasing sequences of real numbers. Then

$$\frac{a_1b_1 + a_2b_2 + \dots + a_nb_n}{n} \geq \frac{a_1 + a_2 + \dots + a_n}{n} \cdot \frac{b_1 + b_2 + \dots + b_n}{n} \geq \frac{a_1b_n + a_2b_{n-1} + \dots + a_nb_1}{n}$$

Theorem 7 (Schur) Let a, b, c be nonnegative reals and $r > 0$. Then

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or some two of a, b, c are equal and the other is 0.

Remark - This can be improved considerably. (See the problems section.) However, they are not as well known (as of now) as this form of Schur, and so should be proven whenever used on a contest.

Theorem 8 (Majorization) Let $f : I \rightarrow \mathbb{R}$ be a convex on I and suppose that the sequence x_1, \dots, x_n majorizes the sequence y_1, \dots, y_n , where $x_i, y_i \in I$. Then

$$f(x_1) + \dots + f(x_n) \geq f(y_1) + \dots + f(y_n)$$

Theorem 9 (Muirhead) Suppose the sequence a_1, \dots, a_n majorizes the sequence b_1, \dots, b_n . Then for any positive reals x_1, \dots, x_n ,

$$\sum_{sym} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \geq \sum_{sym} x_1^{b_1} x_2^{b_2} \dots x_n^{b_n}$$

where the sums are taken over all permutations of n variables.

Remark - Although Muirhead's theorem is a named theorem, it is generally not favorably regarded as part of a formal olympiad solution. Essentially, the majorization criterion guarantees that Muirhead's inequality can be deduced from a suitable application of AM-GM. Hence, whenever possible, you should use Muirhead's inequality only to deduce the correct relationship and then explicitly write all of the necessary applications of AM-GM. For a particular case this is a simple matter.

2 New Techniques

2.1 Notation

This isn't really a technique, but having good notation allows you to write out otherwise difficult algebraic expansions with relative ease. The two major examples of this are the cyclic and symmetric sums, denoted by \sum_{cyc} and \sum_{sym} . In the 3-variable case (the idea should easily expand to other cases),

$$\sum_{cyc} f(a, b, c) = f(a, b, c) + f(b, c, a) + f(c, a, b)$$

$$\sum_{sym} f(a, b, c) = f(a, b, c) + f(a, c, b) + f(b, a, c) + f(b, c, a) + f(c, a, b) + f(c, b, a)$$

2.2 Calculus

Some of the following techniques will be calculus-based. Therefore, before beginning, we will go through a quick crash-course on differentiating.

$\frac{\delta}{\delta x_i} f(x_1, \dots, x_i, \dots, x_n)$ is the partial derivative of f with respect to x_i . It is equivalent to the derivative $\frac{d}{dx_i} f(x_i)$ of f with respect to x_i where we treat $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ as constants. A function is **increasing** if its derivative is positive, **decreasing** if its derivative is negative, and **constant** if its derivative is zero.

The following formulas should allow you to find the derivative of any function you come across:

1. $\frac{d}{dx} e^x = e^x$
2. $f'(x) := \frac{d}{dx} f(x)$
3. (Product Rule) $(fg)'(x) = (f'g + g'f)(x)$
4. (Quotient Rule) $\left(\frac{f}{g}\right)'(x) = \frac{f'g - g'f}{g^2}(x)$
5. (Chain Rule) $\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x)$ or $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$
6. $\frac{d}{dx} (cf(x)) = c \frac{d}{dx} f(x)$
7. $\frac{d}{dx} (f + g)(x) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$
8. (Power Rule) $\frac{d}{dx} x^n = nx^{n-1}$
9. $\frac{d}{dx} a^x = a^x \ln(a)$
10. $\frac{d}{dx} \ln(x) = \frac{1}{x}$
11. $\frac{d}{dx} \sin(x) = \cos(x)$
12. $\frac{d}{dx} \cos(x) = -\sin(x)$

2.3 Convexity

A function is **convex** if its second derivative $\frac{d^2}{dx^2} f(x) = \frac{d}{dx} \frac{d}{dx} f(x)$ is positive, and **concave** if it is negative. In addition to the possible application of Jensen's inequality, if a function is convex then its maximum occurs when all variables are at one of the endpoints of its domain, and its minimum can be found with Jensen (if an equality case exists). The opposite is true for a concave function.

2.4 Isolated Fudging

The idea here is to compare some complicated function $f(x_1, \dots, x_n)$ to a (generic) nicer function. Common choices are linear and quadratic functions, as well as the function $c \frac{x_i^r}{x_1^r + \dots + x_n^r}$. After picking a generic function that you think would be helpful, you can figure out particular values of exponents, coefficients, etc. through the use of calculus. In particular, both the functions and their derivatives must be equal at the equality cases for the inequality to be true. Here is an example:

(IMO 2001, Problem 2) Let a, b, c be positive reals. Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

Here we will compare $\sum_{cyc} \frac{a}{\sqrt{a^2 + 8bc}}$ to $\sum_{cyc} \frac{a^r}{a^r + b^r + c^r} = 1$. One of the equality cases is $a = b = c = 1$, so we set b and c to 1. Then, taking the partial derivative of the first term in both inequalities with respect to a yields $\frac{d}{da} \frac{a}{\sqrt{a^2 + 8}} = \frac{\sqrt{a^2 + 8} - \frac{a^2}{\sqrt{a^2 + 8}}}{a^2 + 8} = \frac{8}{27}$ when evaluated at $a = 1$, and $\frac{d}{da} \frac{a^r}{a^r + 2} = \frac{ra^{r-1}(a^r + 2) - ra^{r-1}a^r}{(a^r + 2)^2} = \frac{2}{9}r$ when evaluated at $a = 1$. Then solving $\frac{2}{9}r = \frac{8}{27}$ we have $r = \frac{4}{3}$. Then we need to show that $\frac{a}{\sqrt{a^2 + 8bc}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}}$, or, equivalently, $a + (\frac{b^4}{a})^{\frac{1}{3}} + (\frac{c^4}{a})^{\frac{1}{3}} \geq \sqrt{a^2 + 8bc}$. Squaring and subtracting a^2 from both sides, we get a new inequality, which should follow directly from weighted AM-GM. This is what should go into your scratchwork. However, a writeup should look something like this:

Consider the following inequality:

$$\sqrt[3]{a^2b^4} + \sqrt[3]{a^2b^4} + \sqrt[3]{a^2c^4} + \sqrt[3]{a^2c^4} + \sqrt[3]{b^8a^{-2}} + \sqrt[3]{b^4c^4a^{-2}} + \sqrt[3]{b^4c^4a^{-2}} + \sqrt[3]{c^8a^{-2}} \geq 8 \sqrt[24]{b^24c^24} = 8bc$$

If we add a^2 to both sides, then this factors to $(a + \sqrt[3]{b^4a^{-1}} + \sqrt[3]{c^4b^{-1}})^2 \geq a^2 + 8bc$, after which it follows that $\frac{a}{\sqrt{a^2 + 8bc}} \geq \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}}$, so that

$$\sum_{cyc} \frac{a}{\sqrt{a^2 + 8bc}} \geq \sum_{cyc} \frac{a^{\frac{4}{3}}}{a^{\frac{4}{3}} + b^{\frac{4}{3}} + c^{\frac{4}{3}}} = 1$$

the desired result.

2.5 Logs

Oftentimes, when given a problem which involves many terms multiplied together instead of added, it can be useful to take the log of the expressions, then solve the new inequality, usually using something involving convexity and concavity.

2.6 Factoring

To aid in factoring, consider that multi-variables polynomials behave like single-variable polynomials in that if, say, the polynomial has a zero at $a = b$, then $(a - b)$ is a root of the polynomial.

3 Problems

Thanks to Thomas Mildorf for assembling the following problems.

1. (MOP 2004) $\sum_{cyc} \left(\frac{a + 2b}{a + 2c} \right)^3 \geq 3$
2. (TMildorf) Given $a_1 + \dots + a_n = 1$, then $a_1a_2 + \dots + a_{n-1}a_n \leq \frac{1}{4}$.
3. (Ukraine 2001) Given $x + y + z = 1$, prove that $ax + by + cz + 2\sqrt{(ab + bc + ca)(xy + yz + zx)} \leq a + b + c$.

4. (Vasile Cirtoaje) $\sum_{cyc} \frac{a^3}{(2a^2 + b^2)(2a^2 + c^2)} \leq \frac{1}{a + b + c}$

5. (Darij Grinberg) $\sum_{cyc} \frac{b^2 + c^2 - a^2}{a(b + c)} \geq \frac{3}{2}$