

TJUSAMO Practice

1. Let B be a point on a circle S_1 , and let A be a point distinct from B on the tangent at B to S_1 . Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B . Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC .
2. Find all functions f from the reals to the reals such that $f(f(x) + y) = 2x + f(f(y) - x)$.
3. Let m be a fixed integer greater than 1. The sequence x_0, x_1, x_2, \dots is defined as follows: $x_i = 2^i$ if $0 \leq i \leq m - 1$, and $x_i = \sum_{j=1}^m x_{i-j}$ if $i \geq m$. Find the greatest k for which the sequence contains k consecutive terms divisible by m .
4. Let $n \geq 3$ be an integer and t_1, t_2, \dots, t_n be positive real numbers such that

$$n^2 + 1 > (t_1 + \dots + t_n) \left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)$$

Show that t_i, t_j, t_k are the side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

5. Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1, and the sum of all entries is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding C in absolute value.