TJUSAMO Problem Session #3: Speed

HMao

March 26th, 2007

Many of you have been complaining that the problems are too hard. Consequently, today we are going to solve some shorter and hopefully easier problems. These problems will mostly consist of relatively few steps, which will allow you to get a grasp at using some techniques that you might not be able to see when other steps are required for a longer problem. Many of these may end up easy to handwave, so make sure your proof is rigorous. The problems are not necessarily in difficulty order, but they are all doable.

1. If $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1$, prove that $a_1 + a_2 + \cdots + a_n + n \leq a_1 a_2 \ldots a_n + 1$.

2. If a, b are naturals and $2a+1$ and $2b+1$ are relatively prime, prove that $\frac{2^{2a+1}+1}{3}$ and $\frac{2^{2b+1}+1}{3}$ are relatively prime.

3. Find all naturals n such that there are n distinct factors of $n!$ which add up to $n!$.

4. If p, m are odd primes, x, y, z are integers, $m - 1 = 2p$, and $x^p + y^p + z^p \equiv 0 \pmod{m}$, prove that $xyz \equiv 0 \pmod{m}$.

5. Prove that n and $\sqrt[3]{2^{2^n} + 1}$ can never both be natural.

6. Prove that n and $\displaystyle\sum_{i=1}^{n} \frac{1}{i}$ can never both be natural.

7. Let $\triangle ABC$ have circumcenter O, and let D and E be points on AB and AC, respectively. If $CBDE$ is a cyclic quadrilateral, prove that AO is perpendicular to DE.

8. 42 children are sitting in a circle. If no five consecutive people are the same gender, find the maximum possible number of girls in the circle.

9. Alice and Bob take turns marking A and B, respectively, on the squares of an infinite checkerboard. A player wins by getting 5 in a row horizontally or vertically (not diagonally). Prove that in a perfect game, neither player will win.

10. Prove that every natural n divides a positive fibonacci number.

11. Prove that $x^{2^{2^{2^{2^2}}}} + 1$ is irreducible.

12. If $a, b, c \geq 2$, prove that $(a^3 + b)(b^3 + c)(c^3 + a) \geq 125abc$.

13. A triangular grid is created by tessellating an equilateral triangle of side length n into n^2 unit equilateral triangles. Count the number of unit rhombuses bounded by the grid.