TUSAMO Practice

Here is a list of all of the decent-looking USAMO problems of the past decade that aren't insane and haven't been covered already. Try to work through as many as you can individually. Also, try to do them in order so that you aren't tempted to skip around to the ones that seem easier. The officers will help people who are completely stuck.

1 Ideas

While working, consider the following:

1. If you are having trouble proving something, then try to disprove it instead.
2. Try doing stupid things before trying clever things.
3. List the major methods that are most likely to work on this problem, and consider what information each one gives you.
4. Make sure you are using all of the information.

2 New Problems

1. (1998/1) The sets $\{a_1, a_2, \ldots, a_{999}\}$ and $\{b_1, b_2, \ldots, b_{999}\}$ together contain all the integers from 1 to 1998. For each i, $|a_i - b_i| \in \{1, 6\}$. Find the last digit of $\sum_{i=1}^{999} |a_i - b_i|$.
2. (1999/1) Certain squares of an nxn board are colored black and the rest white. Every white square shares a side with a black square. Every pair of black squares can be joined by chain of black squares, so that consecutive members of the chain share a side. Show that there are at least $\frac{n^2-2}{3}$ black squares.
3. (2003/1) Show that for each n we can find an n-digit number with all its digits odd which is divisible by 5^n.
4. (2003/4) ABC is a triangle. A circle through A and B meets the sides AC, BC at D, E respectively. The lines AB and DE meet at F. The lines BD and CF meet at M. Show that M is the midpoint of CF iff $(MB)(MD) = MC^2$.
5. (2005/1) Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.
6. (2005/4) Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table? (The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)
7. (2000/5) ABC is a triangle. C_1 is a circle through A and B. We can find circle C_2 through B and C touching C_1, circle C_3 through C and A touching C_2, circle C_4 through A and B touching C_3 and so on. Show that C_7 is the same as C_1.
8. (2000/3) A player starts with A blue cards, B red cards and C white cards. He scores points as he plays each card. If he plays a blue card, his score is the number of white cards remaining in his hand. If he plays a red card it is three times the number of blue cards remaining in his hand. If he plays a white card, it is twice the number of red cards remaining in his hand. What is the lowest possible score as a function of A, B and C and how many different ways can it be achieved?

9. (2006/2) For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of side k has sum at most $\frac{N}{2}$.

10. (1996/5) D lies inside the triangle ABC. $\angle BAC = 50$, $\angle DAB = 10$, $\angle DCA = 30$, $\angle DBA = 20$. Show that $\angle DBC = 60$.

11. (2005/2) Prove that the system

$$x^6 + x^3 + x^3 y + y = 147^{157}$$

$$x^3 + x^3 y + y^2 + y + z^9 = 157^{147}$$

has no solutions in integers x, y, and z.

12. (1997/5) Let $x, y, z > 0$. Show that

$$\frac{xyz}{x^3+y^3+xyz} + \frac{xyz}{y^3+z^3+xyz} + \frac{xyz}{z^3+x^3+xyz} \leq 1$$

13. (1998/5) Show that one can find a finite set of integers of any size such that for any two members the square of their difference divides their product.

14. (1997/4) A sequence of polygons is derived as follows. The first polygon is a regular hexagon of area 1. Thereafter each polygon is derived from its predecessor by joining two adjacent edge midpoints and cutting off the corner. Show that all the polygons have area greater than $\frac{1}{3}$.

15. (1996/2) Let S be a set of n positive integers. Let P be the set of all integers which are the sum of one or more distinct elements of S. Show that we can find n subsets of P whose union is P such that if a, b belong to the same subset, then $a \leq 2b$.

16. (2003/3) Given a sequence S_0 of $n+1$ non-negative integers, $a_0, a_1, \ldots, a_n$ we derive another sequence S_1 with terms $b_0, b_1, \ldots, b_n$, where b_i is the number of terms preceding a_i in S_0 which are different from a_i (so $b_0 = 0$). Similarly, we derive S_2 from S_1 and so on. Show that if $a_i \leq i$ for each i, then $S_n = S_{n-1}$.

17. (2002/5) Show that we can link any two integers m, n greater than 2 by a chain of positive integers $m = a_1 \to a_2 \to \ldots \to a_{k+1} = n$, so that the product of any two consecutive members of the chain is divisible by their sum. (For example, $7, 42, 21, 28, 70, 30, 6, 3$ links 7 and 3.)

18. (2002/6) A tromino is a $1x3$ rectangle. Trominoes are placed on an nxn board. Each tromino must line up with the squares on the board, so that it covers exactly three squares. Let $f(n)$ be the smallest number of trominoes required to stop any more being placed. Show that for all $n > 0$, $\frac{n^2}{7} - hn \leq f(n) \leq \frac{n^2}{5} + kn$ for some reals h and k.

19. (1996/3) Given a triangle, show that we can reflect it in some line so that the area of the intersection of the triangle and its reflection has area greater than $\frac{2}{3}$ the area of the triangle.

20. (1996/4) A type 1 sequence is a sequence with each term 0 or 1 which does not have 0, 1, 0 as consecutive terms. A type 2 sequence is a sequence with each term 0 or 1 which does not have 0, 0, 1, 1 or 1, 1, 0, 0 as consecutive terms. Show that there are twice as many type 2 sequences of length $n+1$ as type 1 sequences of length n.

21. (1999/2) For each pair of opposite sides of a cyclic quadrilateral take the larger length less the smaller length. Show that the sum of the two resulting differences is at least twice the difference in length of the diagonals.

22. (1999/4) A set of $n > 3$ real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.

23. (1997/2) ABC is a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

24. (1998/4) A 98x98 chess board has the squares colored alternately black and white in the usual way. A move consists of selecting a rectangular subset of the squares (with boundary parallel to the sides of the board) and changing their color. What is the smallest number of moves required to make all the squares black?

25. (2000/2) The incircle of the triangle ABC touches BC, CA, AB at D, E, F respectively. We have $AF \geq BD \geq CE$, the inradius is r and we have $\frac{2}{AF} + \frac{5}{BD} + \frac{5}{CE} = \frac{6}{r}$. Show that ABC is isosceles and find the lengths of its sides if $r = 4$.

26. (2000/1) Show that there is no real-valued function f on the reals such that for all $x, y \in \mathbb{R}$,

$$\frac{f(x) + f(y)}{2} \geq f(\frac{x+y}{2}) + |x - y|$$

27. (2000/4) How many squares of a 1000x1000 chessboard can be chosen, so that we cannot find three chosen squares with two in the same row and two in the same column?

28. (1998/2) Two circles are concentric. A chord AC of the outer circle touches the inner circle at Q. P is the midpoint of AQ. A line through A intersects the inner circle at R and S. The perpendicular bisectors of PR and CS meet at T on the line AC. What is the ratio $\frac{AT}{TC}$?

29. (1999/5) Two players play a game on a line of 2000 squares. Each player in turn puts either S or O into an empty square. The game stops when three adjacent squares contain S, O, S in that order and the last player wins. If all the squares are filled without getting S, O, S, then the game is drawn. Show that the second player can always win.

30. (2001/1) What is the smallest number of colors needed to color 8 boxes of 6 balls (one color for each ball), so that the balls in each box are all different colors and any pair of colors occurs in at most one box.

31. (2001/2) The incircle of the triangle PBC touches BC at U and PC at V. The point S on BC is such that $BS = CU$. PS meets the incircle at two points. The nearer to P is Q. Take W on PC such that $PW = CV$. Let BW and PS meet at R. Show that $PQ = RS$.

32. (2001/5) A set of integers is such that if a and b belong to it, then so do $a^2 - a$, and $a^2 - b$. Also, there are two members a, b whose greatest common divisor is 1 and such that $a - 2$ and $b - 2$ also have greatest common divisor 1. Show that the set contains all the integers.

33. (2002/1) Let S be a set with 2002 elements and P the set of all its subsets. Prove that for any n (in the range from zero to $|P|$) we can color n elements of P white, and the rest black, so that the union of any two elements of P with the same color has the same color.

34. (2002/2) The triangle ABC satisfies the relation

$$\cot^2(\frac{A}{2}) + 4\cot^2(\frac{B}{2}) + 9\cot^2(\frac{C}{2}) = \frac{9(a+b+c)^2}{49r^2}$$

where r is the radius of the incircle (and $a = |BC|$ etc, as usual). Show that ABC is similar to a triangle whose sides are integers and find the smallest set of such integers.

35. (2003/2) A convex polygon has all its sides and diagonals with rational length. It is dissected into smaller polygons by drawing all its diagonals. Show that the small polygons have all sides rational.

36. (2003/5) Let $x, y, z > 0$. Prove that

$$\sum_{cyc} \frac{(2x+y+z)^2}{2x^2+(y+z)^2} \leq 8$$

37. (2004/4) Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing in the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number in that row is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from one to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

38. (2005/3) Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct point C_1 in such a way that convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct point B_1 in such a way that convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that points B_1, C_1, P, and Q lie on a circle.

39. (2005/5) Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

40. (2005/6) For m a positive integer, let $s(m)$ be the sum of the digits of m. For $n \geq 2$, let $f(n)$ be the minimal k for which there exists a set S of n positive integers such that $s\left(\sum_{x\in X} x\right) = k$ for any nonempty subset $X \subset S$. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$

41. (2006/1) Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

iff s is not a divisor of $p-1$. ($\{x\}$ denotes the fractional part of x.)

42. (2006/4) Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \ldots + a_k = a_1 \cdot a_2 \cdot \ldots \cdot a_k = n$.

43. (2006/5) A mathematical frog jumps along the number line. The frog starts at 1, then jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$, where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.