

# TJUSAMO08 Practice #2: The Essence of Geometry

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September 27th, 2007

Olympiad geometry is radically different from short-answer geometry. The geometry problems one sees on the AIME all fall under three categories: trivial, 3D, and computationally intensive. On the other hand, most olympiad geometry problems are 2D, difficult, and require little to no computation. In fact, presenting solutions involving even the slightest unnecessary computation is frowned upon. Geometric solutions should be beautiful. Sometimes that means unmotivated and seemingly random, and to tell the truth, a lot of steps one takes in a geometry problem are hard to motivate. This is because geometry is a subject in which practice overrides machinery and intuition reigns supreme. Any awesome geometric solution ought to have its fair share of "how did you see that"'s and "ZORGR!"'s. People see those things either from trying random things (less likely to work), having done a ton of geometry problems and having seen similar tactics in play (more likely to work), or having done the problem or seen the solution before (most likely to work). To get better at geometry, you simply have to do lots of problems. At first people tend to hate geometry. This is naturally caused by the person getting stuck on geometry problems repeatedly from just not knowing what to do. But consider the college process. You have to do all the ridiculous application work first, then when you get into college it's like ORZ NORZ and then you are happy and everything. Well, geometry is just as addictive as senioritis. Once you get accustomed to doing geometry problems you will be hooked. But getting there is the hard part... let's start with the basics

## 1 Circles

A circle is defined as the set of all points that are a positive distance  $r$  from a point  $O$ .  $r$  is called the **radius**.  $O$  is called the **center**. Most geometry problems involving circles also involve a polygon either **inscribed** inside a circle (all its vertices are on the circle), or **circumscribed** around a circle (all its sides are tangent to the circle). The union of a circle and its interior forms a **disk**.

### 1.1 Angles

An **angle** is a measure of the directed distance between two rays. Circles have  $360^\circ$ , or  $2\pi$  radians. The measure of an inscribed angle is half of the length of the arc it intercepts, no

matter where on the circle the vertex lies. The measure of a central angle is equal to the length of its arc.

## 1.2 Power of a Point

To find the **power** of a point  $P$  relative to a circle  $\omega$ , choose any point on the circle  $A$ , and draw line  $PA$ . Let  $B$  be the point of intersection of  $PA$  with  $\omega$  other than  $A$ . If  $A$  is the only point of intersection, the power is  $PA^2$ . Otherwise the power is  $PA * PB$ . The power of a point with respect to a given circle is constant. Also, the point can be inside, outside, or on the circle. Note that a point on the circle will have a power of 0 with respect to that circle.

## 1.3 Cyclic Quadrilaterals

A **cyclic quadrilateral** is a polygon with four sides that can be inscribed in a circle. All cyclic quads are convex, since all their angles are less than  $180^\circ$ . Opposite angles in a cyclic quad add up to  $180^\circ$ . The converse is also true: if a quadrilateral's opposite angles add up to  $180^\circ$ , then it is cyclic. This is the easiest way to find a cyclic quad. Cyclic quads are usually tools used in angle chasing, and sometimes, proving a quadrilateral is cyclic will be the key step in a problem.

# 2 Triangles

A triangle is defined by three non-collinear points. Triangles are the simplest polygons, and also the most important ones. Over half of the geometry problems you'll see are based on a triangle (probably  $\triangle ABC$ ). There are several key triangle terms one must recognize.

Let  $\triangle ABC$  be a triangle. Its vertexes are  $A$ ,  $B$ , and  $C$ . Its angles are  $\angle A$ ,  $\angle B$ , and  $\angle C$ . Its sides are  $AB$ ,  $BC$ , and  $AC$ .

- **cevian**: A line segment connecting a vertex with a point on the opposing side.
- **median**: A line segment connecting a vertex with the midpoint of the opposing side.
- **centroid**: The intersection point of the three medians.
- **incircle**: A circle inscribed within a polygon. Its center is the **incenter**.
- **excircle**: A circle escribed about a triangle. It is tangent to one side as well as the extensions of other two sides. Its center is the **excenter**.
- **circumcircle**: A circle circumscribed around a polygon. Its center is the **circumcenter**.
- **altitude**: A line segment connecting a vertex with a point on the opposite side and perpendicular to that opposite side.

- **orthocenter:** The point at which the three altitudes intersect. This is not the center of an "orthocircle".

## 2.1 Similar Triangles

When two triangles are **similar**, all their corresponding angles are equal and all their corresponding side lengths are in the same ratio. When two triangles are **congruent**, all their corresponding side lengths and angles are equal. There are multiple ways to prove two triangles are similar/congruent.

- **AA similarity:** two pairs of angles are equal. Clearly the last pair is also equal.
- **SAS similarity:** two pairs of side lengths have the same ratio, and the angle between the two sides is the same in both triangles. If both pairs of sides are equal, then the triangles are congruent.
- **SSS similarity:** all corresponding side lengths have the same ratio. If all corresponding side lengths are the same, then the triangles are congruent.
- **SSA dissimilarity:** if two corresponding side lengths have the same ratio and one of the angles not between the two sides is the same for both triangles, the triangles are NOT necessarily similar!

## 2.2 Angle Chasing

Now that we are armed with tools, we can start angle chasing. This is the most important skill a geometer must learn. Here are some tips for how one should go about angle chasing.

- Angle chasing is not just making trivial observations such as: "hey, these two angles are vertical, they must be the same!" Instead, you must use similar triangles, cyclic quads, angle bisectors, etc to discover how angles are related. Angle chasing can be difficult!
- Figure out what you want to chase and what you want to end up with. Don't angle chase purposelessly.
- Try splitting an angle into two different angles that you have more knowledge of.
- If you know two angles are equal or supplementary, try making a cyclic quad out of them.
- Introduce a few parameters and identify which angles can be easily expressed in terms of those parameters. If the whole diagram can be expressed in terms of one or two parameters, you've pretty much solved the problem.
- SPIRAL SIMILARITY!(Ignore if you are confused or indifferent.)

### 3 Theorems

This section has been purposely omitted. Focus on the fundamentals today! Angle chasing is a THE MOST VALUABLE skill for a geometer. You will not need to know any fancy theorems for now. Angle chasing will be enough of a challenge in itself.

### 4 The Art of Geometry Problem Solving

Among the fields of olympiad problem solving, geometry is the odd one out. Methods for tackling geometry problems differ from the norm. Here are some tips to get you oriented:

- Write down all the givens of the problem in their simplest forms. Each of these facts will be used in one way or another, often multiple times.
- Write down what you're trying to prove and facts that lead directly to this end result.
- Try starting from your result, and work backwards. Try to get closer to your givens.
- Keep separate tabs of what you've figured out working forwards and working backwards.
- If you get stuck, look back at your givens. Chances are that you haven't used one of them. Try to figure out how to use it.
- Draw a big, neat, clean diagram. This cannot be stressed enough. More on this later.
- Don't start by brute forcing with coordinates or complex numbers. This might give you a headache in the long run, and oftentimes won't work. Instead, try to identify the key ideas and elements of the problem.
- Don't let the problem intimidate you. When looking at a complicated problem, don't think: "ugh, I probably don't know the necessary background machinery to find a solution, so I'll try brute forcing." Geometry problems usually don't require much complicated knowledge. Try to play around with the problem until something dawns on you. This way, if you have a chance of solving it, you don't waste time thinking that you don't.
- The only way you'll ever get good at geometry is by doing problems. You must have seen enough methods and results so that you are familiar with any problem you are given.
- After you solve a problem, try to look back at your solution and learn from it. If you are proud that you solved a difficult problem, identify the obstacles you overcame and remember how you overcame them. Use similar tactics in the future. You must develop your own strategies and intuitions.

## 4.1 Diagrams

"Hah a diagram, I don't need a diagram!" You can't be like that guy. You need a diagram. Always start off a problem by drawing a good diagram.

- Always draw big. Give enough space to make markings and constructions.
- Draw your diagram multiple times until you are satisfied with it. This is well worth the time and effort.
- Make sure all the givens look somewhat to scale in your diagram.
- Draw multiple cases to familiarize yourself with the problem, such as drawing an acute triangle and an obtuse triangle. Then focus on one case at a time.
- Don't clog up the diagram with needless lines or repetitive markings. By keeping the diagram clear, you'll keep your mind clear. Don't make constructions without motivation. If your diagram gets messy, draw a new one. If your paper gets messy, use a blank sheet.
- When you attempt to draw a polygon inscribed in a circle, draw the circle first. Compasses help.
- Draw straight lines. A straightedge will help.
- Often drawing diagrams to scale will make things stand out, such as showing that three points could be collinear. This will hint you at possible intermediate steps in your proof.
- If a problem introduces a generic triangle, try to make it as scalene as possible. A  $45^\circ$ - $60^\circ$ - $75^\circ$  triangle usually does the trick.

## 5 Two Roads Diverged in a Yellow Wood...

### 5.1 The Road Not Taken

1. Square  $EFGH$  is inside the square  $ABCD$  so that each side of  $EFGH$  can be extended to pass through a vertex of  $ABCD$ . Square  $ABCD$  has side length  $\sqrt{50}$  and  $BE = 1$ . What is the area of the inner square  $EFGH$ ? (A. 25, B. 32, C. 36, D. 40, E. 42)
2. In quadrilateral  $ABCD$ ,  $BC = 8$ ,  $CD = 12$ ,  $AD = 10$ , and  $\angle A = \angle B = 60^\circ$ . Given that  $AB = p + \sqrt{q}$ , where  $p$  and  $q$  are positive integers, find  $p + q$ .
3. Let  $\triangle ABC$  be acute-angled, and let  $P$  and  $Q$  be two points on its side  $BC$ . Construct a point  $C_1$  in such a way that the convex quadrilateral  $APBC_1$  is cyclic,  $QC_1 \parallel CA$ , and the points  $C_1$  and  $Q$  lie on opposite sides of the line  $AB$ . Construct a point  $B_1$  in such a way that the convex quadrilateral  $APCB_1$  is cyclic,  $QB_1 \parallel BA$ , and the points  $B_1$  and  $Q$  lie on opposite sides of the line  $AC$ . Prove that the points  $B_1$ ,  $C_1$ ,  $P$ , and  $Q$  lie on a circle.

4. Let  $ABC$  be a triangle. Choose a point  $D$  in its interior. Let  $\omega_1$  be a circle passing through  $B$  and  $D$  and  $\omega_2$  be a circle passing through  $C$  and  $D$  so that the other point of intersection of the two circles lies on  $AD$ . Let  $\omega_1$  and  $\omega_2$  intersect side  $BC$  at  $E$  and  $F$ , respectively. Denote by  $X$  the intersection of  $DF$ ,  $AB$  and  $Y$  the intersection of  $DE$ ,  $AC$ . Show that  $XY \parallel BC$ .
5. In  $\triangle ABC$  the bisector of  $\angle BCA$  intersects the circumcircle again at  $R$ , the perpendicular bisector of  $BC$  at  $P$ , and the perpendicular bisector of  $AC$  at  $Q$ . The midpoint of  $BC$  is  $K$  and the midpoint of  $AC$  is  $L$ . Prove that the triangles  $RPK$  and  $RQL$  have the same area.

## 5.2 The Road You Shall Take

1. Prove that the sum of the angles in a triangle is  $180^\circ$  without using any information related to a triangle or a circle. (ie using axioms of Euclidean geometry if you know those)
2. Let  $D$  be a point in the interior of  $\triangle ABC$ , and let  $E, F, G$  be points on  $AB, BC, CA$ , respectively. Given that  $AEBD$  and  $BFCD$  are cyclic, prove that  $CGAD$  is also cyclic.
3. Let  $ABC$  be a triangle with orthocenter  $H$  and circumcenter  $O$ . Prove that  $\angle HAO = |\angle B - \angle C|$ .
4. Given a compass and straightedge and a natural  $n$ , split a given line segment into  $n$  equal parts.
5. Let  $AB$  and  $CD$  be two lines which intersect at  $M$ . Given that the circumcircles of  $\triangle ACM$  and  $\triangle BDM$  meet at  $M$  and  $N$ , prove that  $\triangle ABN$  is similar to  $\triangle CDN$ .
6. Let  $M$  and  $N$  be the midpoints of sides  $AD$  and  $BC$  of rectangle  $ABCD$ , respectively. Let  $P$  be a point on ray  $CD$  but not on side  $CD$ . Let point  $Q$  be the intersection of  $AC$  and  $PM$ . Prove that  $\angle MNQ = \angle MNP$ .
7. Let  $ABC$  be a triangle. Let  $D$  be the midpoint of side  $BC$ . Let  $E$  be a point on  $BC$  such that  $AE$  is the angle bisector of  $\angle BAC$ . The circumcircle of  $DEA$  intersects  $AB$  at  $X$  and  $AC$  at  $Y$ . Prove that  $BX = CY$ .
8. (ZAbel) Let  $AD$  be the angle bisector of  $\triangle ABC$ . Let  $E, F$  be points on lines  $AB, AC$ , respectively, such that  $ABDE$  and  $ACDF$  are cyclic with circumcenters  $O_B$  and  $O_C$ , respectively. Given that the points  $B, C, O_B, O_C$  are concyclic, find all possible values of  $\angle BAD$ .