TJUSAMO #4: Problem Session/Review

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The goal of this practice is to review the previous two practices as well as to get experience working olympiad-style problems.

1 Warm-up

Prove that there are infinitely many primes 1 mod 4. (Hint: If $p|n^2+1$ and n is even then $p \equiv 1 \pmod 4$.)

2 Geometry

Uh...since Haitao's not here and I can't do geometry, I just stole a problem from the IMO shortlist that I could figure out how to solve.

1. (IMO SL 2003/1) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

3 Number Theory

1. (USAMO 2006/1) Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

iff s is not a divisor of $p-1$. ($\{x\}$ denotes the fractional part of x.)

2. Let S_p denote the set of all permutations on p elements, and let $f : S_n \to \mathbb{Z}/p\mathbb{Z}$ be defined as

$$f(\pi) = a_1\pi(1) + a_2\pi(2) + \ldots + a_p\pi(p)$$

where $\pi(1), \ldots, \pi(p)$ denote the images of $1, \ldots, p$ under the permutation π. Prove that either $a_1 = a_2 = \ldots = a_n$ or all non-zero residues mod p are attained by f. For this problem you may assume that there exists some residue such that $ord_p(r) = p-1$.

4 General Problems (aka Combinatorics)

1. (Hungary 2007) If 51 positive integers have a sum of 100, prove that they can be divided into two subsets each with a sum of 50.

2. Prove that any set of n integers has a subset whose sum is divisible by n.

3. (Yan-Quan Feng) Let S be a subset of $\mathbb{Z}$ such that the overall gcd of S is 1 and such that $s \in S \iff -s \in S$. Suppose that for every bijective function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(0) = 0$ and $f(x+s) = f(x) + s'$ for some $s' \in S$ whenever $s \in S$ and $x \in \mathbb{Z}$, it also is true that $f(s_1 + s_2) = f(s_1) + f(s_2)$ for all $s_1, s_2 \in S$. Prove that $f(a+b) = f(a) + f(b)$ for all $a, b \in \mathbb{Z}$.

5 Something to keep you busy if you're done...

Prove that there exists a prime between 1 and n^n that is 1 mod n.