TJUSAMO08 Practice #8: Problem Session #5

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Today our goal is to learn how to solve a really hard problem. In 2004 the International Mathematics Olympiad problem setting committee deemed this problem too hard to put on the contest. This problem is excellent because its solution incorporates all four areas of olympiad problem solving: algebra, geometry, combinatorics, and number theory.

1 The Problem

{IMO SL 2004 N7} Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct lattice points lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of these eight points such that the squares of its side lengths are integers divisible by p^{n+1}.

OK, spend some time bashing your brain against the problem. Play around with the scenario. Try as many different things as you can.

2 Geometry

Prove that a triangle with side lengths a, b, c, circumdiameter d, and area S satisfies the following two identities:

$$2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 = 16S^2$$

$$a^2b^2c^2 = 4S^2d^2$$

3 Important observations

First note that the square of the distance between any two lattice points is an integer, so we will be working with squared edge lengths so we can incorporate number theoretic ideas. Prove that for any triangle with vertices at three of our eight lattice points, if none of the squared edge lengths is divisible by p^{n+1}, exactly one of the squared edge lengths is not divisible by p, and the other two squared edge lengths are divisible by p^n.

Now prove that every quadrilateral with vertices at four of our eight lattice points has an edge whose squared edge length is divisible by p^{n+1}.

4 Woah

Let's make a graph!

Think about how to make a graph out of this problem.

5 Fun part

Now we get to play with our graph. Let's finish up the problem.

6 Additional problem

Do this if you solve the main problem.

{IMO SL 2004 N5, IMO 2004 #6} Find all positive integers that have a multiple whose decimal digits are alternatively odd and even.