Inequalities Problem Set

1 Problems

1. $(a+b)(b+c)(c+a) \geq 8abc$
2. Given $abc = 1$, then $a+b+c \geq 3$.
3. $a^3 \geq 3a - 2$.
4. $(a^2+1)(b^2+1) \geq (a+b)^2$
5. $a^3 + b^3 + c^3 + 3abc \geq a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2$.
6. $\sum_{sym} a^5b^2c \geq \sum_{sym} a^3b^3c^2$.
7. (Tony Liu) $x^2 + \frac{2}{x} \geq 3$.
8. Given $abc \geq a+b+c$, then $abc \geq 3\sqrt{3}$.
9. (Titu Andreescu) $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$.
10. $(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$.
11. (USAMO 04/5) $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + c) \geq (a+b+c)^3$.
12. (MOP) Given $a_1 + \ldots + a_n = 1$, then $\prod_{i=1}^{n} \frac{1 - a_i^k}{a_i^k} \geq (n^k - 1)^n$ for any positive integer k.
13. (Aaron Pixton) Given $abc = 1$, then $5 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq (1+a)(1+b)(1+c)$.
14. Given $a_1 \ldots a_n = 1$, then $\frac{1}{1+a_1} + \ldots + \frac{1}{1+a_n} \leq \frac{a_1 + \ldots + a_n + n}{4}$.

2 More Problems

1. Let $P(x)$ be a polynomial with positive coefficients. Prove that if $P(1) \geq 1$, then $P(\frac{1}{x}) \geq \frac{1}{P(x)}$ for all $x \geq 0$.
2. (USAMO 77/5) Prove that if $0 < p \leq q$, then

$$(a+b+c+d+e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$$

   and determine when equality holds.
3. (IMO 74/5) If a, b, c, d are positive reals, determine the possible values of

$$\frac{a}{a+b+d} + \frac{b}{b+c+a} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

4. Show that

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2}\geq a+b+c$$

5. (USAMO 97/5) Prove that

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{c^3+a^3+abc}{\leq}\frac{1}{abc}$$

6. (Moldova 1999) Show that

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)}\geq\frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c}$$

7. (Tuymaada 2000) Prove that for all reals $0<x_1,\ldots,x_n\leq\frac{1}{2}$,

$$\left(\frac{n}{x_1+\ldots+x_n}-1\right)^n\leq\prod_{i=1}^{n}\left(\frac{1}{x_i}-1\right)$$