TJUSAMO Contest #1 **Solutions**

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1

Given a positive integer n, prove that there exists an arithmetic sequence of length n that contains all distinct integers, but no perfect squares.

1.1 Algebraic Solution

The integers from n^2+1 to n^2+n from an arithmetic sequence and are all between n^2 and $(n+1)^2$, which are two consecutive squares, so no term of our sequence can be a square.

1.2 Number Theoretic Solution

All the numbers that are two more than a multiple of three cannot be squares, and form an infinitely long arithmetic sequence, so we can take any subsequence of length n that satisfies the problem statement.

1.3 Trivial Solution

By the trivial inequality, $x^2 \geq 0$, so the sequence $-1, -2, \ldots, -n$ has no squares.

1.4 Notes

This problem was pretty trivial.

2

For any odd integer a that is greater than 5 and not divisible by 5:
a) Prove that $a^8 + 98a^4 + 1$ is divisible by 100.
b) Prove that $\frac{a^8+98a^4+1}{100}$ cannot be a prime integer.

2.1 Solution

Note that a is relatively prime to 10. By Euler's Theorem, $a^{\phi(10)} = a^4 \equiv 1 \pmod{10}$, so $10|a^4-1$, so $100|(a^4-1)^2$, so $100|(a^4-1)^2+100a^4$, which leads to our desired result. For the second part, we conjure the factorization $a^8+98a^4+1 = (a^4+8a^2+1)^2-(4a^3-4a)^2 = (a^4+4a^3+8a^2-4a+1)(a^4-4a^3+8a^2+4a+1)$. Now we must prove that each of those factors is greater than 100, so when divided by 100, each factor is still greater than 1. Some algebra gives us $a^4+4a^3+8a^2-4a+1 = a^2(a+2)^2+(2a-1)^2 > 35^2+9^2 > 10^2$ and $a^4-4a^3+8a^2+4a+1 = a^2(a-2)^2+(2a+1)^2 > 15^2+11^2 > 10^2$. Thus we have shown that a^8+98a^4+1 is the product of two integers greater than 100, so $\frac{a^8+98a^4+1}{100}$ is the product of two integers both greater than 1, so it must be composite.

2.2 Motivation

This was a fairly difficult factoring problem. As a general guideline, always try difference of squares first when you want to factor stuff. Think where the number 98 could come from. Your first instinct would probably be that it's 100-1-1, but that is too few terms, so the 100 must be broken up somehow. $100 = 8^2+6^2$ is fairly natural, so we could try putting 8's and 6's as the a^2 term of one of the factors, and playing around with that would get us the elusive factorization.

3

Let P be a point inside $\triangle ABC$. Let D, E, F be points on the sides BC, CA, AB, respectively, such that no two points of the points labelled so far are the same. Given that P, F, A, E are concyclic(points that can be drawn on the same circle), and P, D, B, F are concyclic, prove that P, E, C, D must also be concyclic.

3.1 Solution

Remember that the opposite angles of a cyclic quadrilateral are supplementary. Now we angle chase: $\angle PEC = 180° - \angle PEA = \angle PFA = 180° - \angle PFB = \angle PDB = 180° - \angle PDC$, so PDCE is a cyclic quadrilateral.

3.2 Notes

Yes, even YOU can solve geometry problems. As a general guideline, don't ignore a problem just because it's the last one on the contest or because it's geometry.