TJUSAMO Contest #2 **Solutions**

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Note that all problems in this contest had particularly elegant solutions, though this will usually not be the case on the actual USAMO.

1. Prove that for all integers $n > 10$, $(n)^{2n} > (2n)!$.

Solution: AM-GM on the sequence $\{2n-2, 2n-3, 2n-4, \ldots, 2\}$ yields $n^{2n-3} \geq (2n-2)!$, so $n^{2n} \geq n^3(2n-2)! \geq (2n)(2n)(2n-2)! \geq (2n)!$.

2. Given a cyclic pentagon $ACEBD$ with center O such that A, B, O are collinear, $AB \perp CD$, and AE bisects CO, prove that DE bisects BC.

Solution: Since $\angle ACO = \angle CAO = \angle BCD = \angle BDC$, we have $\triangle ACO \sim \triangle CDB$. Let X be the intersection of AE and CO and let Y be the intersection of CB and ED. Since $\angle CAX = \angle CDY$, we have $\triangle CAX \sim \triangle CDY$. Using our two similarities, $\frac{YC}{BC} = \frac{YC}{DC}\frac{DC}{BC} = \frac{XC}{AC}\frac{AC}{OC} = \frac{XC}{OC} = \frac{1}{2}$, so $CY = BY$.

3. Given a triangle with side lengths a, b, c, prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$$

Solution: WLOG a is the longest side. $a^2b(a-b) + b^2c(b-c) + c^2a(c-a) = a(b-c)^2(b+c-a) + b(a-b)(a-c)(a+b-c) \geq 0$. gg.