IMO Power Round

VMT Math Team

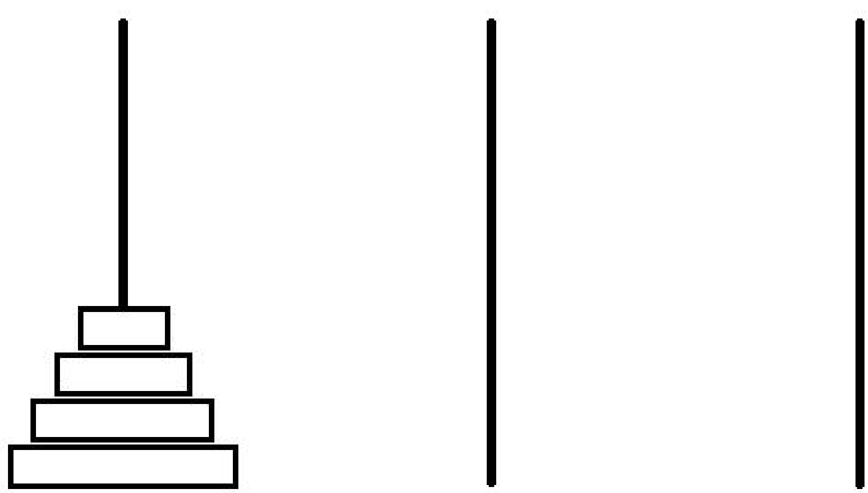
November 13, 2004

Directions - Answer all of the numbered problems. Credit will be based on completeness, clarity, and accuracy. Partial credit will be awarded for incomplete solutions that make progress towards a full solution. A correct answer without accompanying work will receive only partial credit. Incorrect answers will not lose points.

Introduction - The classic puzzle *Towers of Hanoi* is the theme of this round. Our version of the game is as follows: There are three poles (call them left, center, and right) and a stack of n discs. All of the discs are of different sizes and have a hole through their center which allows them to be stacked on the poles. Initially, all of the discs are stacked in a pyramid shape on the left pole, with the largest at the bottom and the smallest at the top. (For example, shown below is this initial setup where $n = 4$.)

We call the arrangment of the discs on the poles a *position.* A position is called *valid* if for any two discs on the same pole, the larger disc is below the smaller one. (So the initial setup is a valid position.) A *move* is defined to be a transition between two valid positions in which the topmost disc of one pole is moved to the top of the stack of another pole.

For this contest, the objective of the puzzle is to move all of the discs to the rightmost pole with as few moves as possible.



The Problems. Each question is worth at most 8 points.

Section I - The problems this section assume the rules described on page 1.

1. How many moves are required to finish the game when $n = 3$? Describe the necessary moves or illustrate this process.

2. How many moves are necessary to finish the puzzle when $n = 4$? Prove your answer.

3. Let a_n denote the least number of moves required to complete the puzzle for n discs. Write an equation relating a_n and a_{n+1}. Justify your answer.

4. Show that 31 moves are required for $n = 5$.

5. Determine, with proof, a formula for a_n in terms of n.

Section II - The problems in this section assume the rules described on page 1 with the additional restriction that in each move a disc can be moved only between adjacent poles - moving a disc from the left pole to the right pole or vice versa in a single move is forbidden.

1. How many moves are required to complete the puzzle when $n = 2$? Describe the necessary moves or illustrate this process.

2. How many moves are needed to finish the game when $n = 3$? Justify your answer.

3. Let b_n denote the least number of moves required to complete the puzzle for n discs. Write an equation relating b_n and b_{n+1}. Justify your answer.

4. Show that 80 moves are required for $n = 4$.

5. Determine, with proof, a formula for b_n in terms of n.

Section III - These problems assume the rules described on page 1 with the modification that instead of three poles there is now another pole added to the right. The goal is to move the sorted stack of discs from the leftmost pole to the rightmost pole. (Call the poles leftmost, left, right, and rightmost.)

1. How many moves are required for $n = 4$? Describe the moves or illustrate such a proceedure.

2. How many moves are required for $n = 5$? Justify your answer.