Impartial Games

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December 6, 2013

Introduction

A *combinatorial game* is a two-player game with the following properties:

- Players alternate taking turns.
- There are deterministic rules that specify which moves can be made.
- There is *perfect information*: both players know everything about the state of the game and the rules.
- The game ends when one player can't make a move. This player either wins or loses.
- Every game ends. (Given an initial position, the number of turns the game can have is bounded)

Today, we'll focus our discussion on games in which the rules that define the game do not depend on the player making a move. We'll call these games *impartial*. An example of a game that is *combinatorial* but not *impartial* is chess: the player that goes first cannot move the black pieces.

Positions

We can define two types of positions in impartial games: winning positions and losing positions. A position is a winning position if the player who has the position can guarantee a win. Similarly, a position is a losing position if the opposing player can guarantee a win. This leads to two rules to identify winning/losing positions:

- Every position that can be reached in a single move from any losing position is a winning position.
- There is at least one losing position that can be reached in a single move from any winning position.

Based on the rules of the game, the terminal positions, or positions from which no move can be made, can be classified as either winning or losing. From this information, we can make a unique classification of each position as winning or losing.

Note that this classification also describes the winning strategy for a winning position: each turn, identify the (guaranteed to exist) losing position that can be reached in one move, and make that move.

Nim

The most well-known impartial game, and the one we'll study first, is the game of Nim. The game of Nim is played with k heaps of stones. The two players alternate taking one or more stones from a single pile. The player who takes the last stone wins the game.

We denote a Nim position consisting of k heaps with sizes $a_1, a_2, \ldots, a_k$ by the tuple $(a_1, a_2, \ldots, a_k)$.

In a Nim game with one heap, every position is a winning position, as the player can simply take all of the stones in the heap and win the game.

A Nim game with two heaps is more complicated. Positions of the form (a, a) for some nonnegative integer a are losing positions, and all other positions are winning positions. We can easily verify this: the terminal position $(0, 0)$ is a losing position, by definition. Given any position (a, b), with $a \neq b$, it is possible to reach the position $(\min(a, b), \min(a, b))$. From any position (a, a), every valid move generates a position (a, b) or (b, a) for some $b < a$.

Examples

1. Determine the winning and losing positions for Nim games of the form $(a, b, 2)$.
2. A two player game is played on a 4 by 4 grid. Players take turns moving a token either any number of spaces to the right, or to the leftmost space in the row above. Classify every position.
3. Same as above, but the token can only be moved one or two spaces to the right (or to the leftmost space in the row above).
4. Generalize #3 to grids of size n by $3m + 1$.
5. Generalize #3 to all grids.

Multiple Heaps

Let $a \oplus b$ denote the bitwise exclusive-or (xor) of a and b: bit k of $a \oplus b$ is zero if bit k of a and b are equal, and one otherwise.

Consider the k-heap Nim position $(a_1, a_2, \ldots, a_k)$. If the sum $a_1 \oplus a_2 \oplus \ldots \oplus a_k$ is zero, it is a losing position. Otherwise, it is a winning position.

The proof of this statement is very similar to the proof for two heap Nim. The terminal position $(0, 0, \ldots, 0)$ is a losing position, and its bitwise xor is 0.

Consider any position $(a_1, a_2, \ldots, a_k)$ with $a_1 \oplus a_2 \oplus \ldots \oplus a_k = n \neq 0$. Written in base 2, n has a leading 1. There must exist an i such that the binary representation of a_i has a 1 in the same position. Since $n \oplus a_i < a_i$, we can take stones from heap i such that it has $n \oplus a_i$ stones left. The result is a position with bitwise xor $n \oplus a_i \oplus (n \oplus a_i) = 0$.

Any move from a position with bitwise xor 0 will flip one of the bits in the result, giving a position with nonzero bitwise xor.

Sprague-Grundy Theorem

Given any two positions A and B of impartial games, let $A + B$ be the position in the combined game such that in each turn, the player can make a move in either (but only one) of the two games. We call two positions G and G' equivalent (denoted $G \sim G'$) if for every game position H, $G + H$ and $G' + H$ are either both winning or both losing positions.

The Sprague-Grundy Theorem states that any position of an impartial game is equivalent to a single Nim heap of some size.

Call this heap size the *nimber* of the game.

The *m*inimum *ex*cluded ordinal of a set of ordinals S, denoted $\text{mex}(S)$, is the smallest ordinal not included in S. For example, $\text{mex}(\{0, 1, 2\}) = 3$, $\text{mex}(\{0, 2, 3\}) = 1$, and $\text{mex}(\{1, 2, 3\}) = 0$.

It can be shown (by structural induction over the space of possible games) that a position G has a nimber equal to the minimum excluded ordinal of the set of nimbers of positions that can be reached from it in a single move. Wikipedia's article on the Sprague-Grundy theorem does a good job of explaining the proof, if you're interested.

Problems

1. Nim is sometimes played with slightly different rules, in which the player forced to take the last stone loses. Completely classify every position of this game.
2. Two players play a game with a number of coins. They take turns flipping either one or two of the coins from heads to tails. The last player to make a move wins. Given the initial positions of the coins, determine the winner if both players play optimally.
3. Two players play a game with finitely many markers placed on the nonnegative half of the number line. Each turn, a player moves one of the markers to an unoccupied spot to the left (toward 0), without crossing over any other markers. The game ends when one player cannot make a move, and this player loses. Given the initial positions of the markers, determine the winner if both players play optimally.
4. Two players write the numbers from 1 to n on a blackboard. They then take turns erasing a number and all powers of it from the board. The person to erase the last number wins. Given n, determine the winner if both players play optimally.
5. Two players take turns marking a single square on an n by 2 grid. Once a square has been marked, it is not allowed to mark that square or the three neighboring squares in the other column. Equivalently, it must be possible at any point to draw a curve from the top of the grid to the bottom that only passes through unmarked squares. The last player to mark a square wins. Given n and the already marked squares, determine the winner if both players play optimally.